A construction of the fundamental solution for a Schrödinger equation with a time dependent potential

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(joint work with P.Antonelli, P.Marcati)

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Presentation of the problem

\[\begin{cases} 
-i \partial_t u(t, x) - \frac{1}{2} \Delta u(t, x) + V(t, x) u(t, x) = 0, \\
u(s, x) = \varphi(x),
\end{cases}\]  

\[(1)\]

\begin{itemize}
  \item[(V-I)] \(V(t, x)\) is a measurable function of \((t, x) \in \mathbb{R} \times \mathbb{R}^n\) and for almost every \(t \in \mathbb{R}\), \(V(t, \cdot) \in C^\infty(\mathbb{R}^n)\).
  \item[(V-II)] \(V \in L^2_t L^\infty_{loc,x}\).
  \item[(V-III)] For any \(|\alpha| \geq 2\), \(\partial_x^\alpha V \in L^2_t L^\infty_x\).
\end{itemize}

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Motivation: two-fluid model

Figure: Specific Heat Capacity
Motivation: two-fluid model (simplified version)

\[
\begin{align*}
\partial_t \rho_s + \text{div} J_s &= 0 \\
\partial_t (J_s) + \text{div} \left( \frac{J_s \otimes J_s}{\rho_s} \right) + \nabla P_s(\rho_s) &= \frac{1}{2} \rho_s \nabla \left( \frac{\Delta \sqrt{\rho_s}}{\sqrt{\rho_s}} \right) - (J_s - Q v_n) \\
\partial_t \rho_n + \text{div}(\rho_n v_n) &= 0 \\
\partial_t (\rho_n v_n) + \text{div}(\rho_n v_n \otimes v_n) + \nabla P_n(\rho_n) &= \eta \Delta v_n + \frac{\eta}{3} \nabla \text{div} v_n,
\end{align*}
\]

1. \(\rho_s, J_s\): superfluid mass and current density
2. \(\rho_n, v_n\) the mass density and the velocity field for the normal fluid
3. \(P_s\) and \(P_n\): self-consistent pressure terms
4. \(\eta\) is the viscosity in the equation for normal fluid
5. \(Q = -(\Delta)^{-1} \nabla \text{div}\).
Superfluid part at NLS level

\[ i\partial_t \psi = -\frac{1}{2} \Delta + \tilde{V} \psi + f(|\psi|^2)\psi, \]

where \( \tilde{V} \) is s.t. \( \nabla \tilde{V} = -Q v_n \in L^2_t L^6_x \).

**Theorem (Ortner, Süli, 2012)**

\( \tilde{V} = V_\infty + V_p \), where

- for a.e. \( t \in \mathbb{R} \), \( V_\infty \in C^\infty(\mathbb{R}^3) \).
- \( V_p \in L^2_t W^{1,6}_x \) and \( \| V_p \|_{L^2_t W^{1,6}_x} \leq \| \nabla \tilde{V} \|_{L^2_t L^6_x} \leq \| v_2 \|_{L^2_t L^6_x} \).
- \( \| \partial^\alpha V_\infty \|_{L^2_t L^\infty_x} \leq C \| \nabla \tilde{V} \|_{L^2_t L^6_x}, \) for every \( |\alpha| \geq 1 \).
The potential $V(t, x)$ grows subquadratically in space, hence it cannot be considered as a Kato type perturbation of the free Laplacian.
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Strategy to construct the fundamental solution to (1): construct directly the integral operator by means of Feynman’s path integrals.
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Strategy to construct the fundamental solution to (1): construct directly the integral operator by means of Feynman’s path integrals.

The solution to (1) is given by an oscillating integral operator (called PARAMETRIX) whose phase is given by the classical action and the integral is performed over a suitable set of paths.
Strategy: "time slicing approximation"

- Let $[s, t]$ an arbitrary interval in $(-T, T)$.
- Oscillatory integral operator

$$E(t, s)\varphi(x) = \frac{1}{(2\pi i (t-s))^\frac{n}{2}} \int_{\mathbb{R}^n} e^{iS(t,s,x,y)} \varphi(y) dy$$

- Define $S(t, s, x, y) \implies$ CLASSICAL ACTION.
- $\Delta: \; s = t_0 < t_1 < \cdots < t_L = t$

$$E(\Delta | t, s) = E(t, t_{L-1})E(t_{L-1}, t_{L-2}) \cdots E(t_1, s).$$

Integral kernel

$$I(\Delta | t, s, x, y) = \prod_{j=1}^{L} \left( \frac{1}{2\pi i (t_j - t_{j-1})} \right)^\frac{n}{2}$$

$$\times \int_{\mathbb{R}^n} \cdots \int_{\mathbb{R}^n} \prod_{j=1}^{L} \exp \left\{ \sum_{j=1}^{L} S(t_j, t_{j-1}, x_j, x_{j-1}) \right\} \prod_{j=1}^{L} dx_j$$

If $\delta(\Delta) = \sup_{j=1,\ldots,L} |t_j - t_{j-1}| \to 0 \implies E(\Delta | t, s) \to U(t, s)$. 

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- $V \in L_t^\infty L^\infty_{x,\text{loc}}$ (locally boundedness in time)
- $\partial_x^\alpha V \in L^\infty(\mathbb{R} \times \mathbb{R}^n)$ for any $|\alpha| \geq 2$

\[\Downarrow\]

Convergence of $E(\Delta|t, s)$ to the propagator for (1).
**REAL NOVELTY** The potential is not locally bounded in time: presence of singularities in time, which are $L^2$ however.

- Trajectories are not Lipschitz continuous but only Hölder continuous with exponent 1/2.
- The classical action inherits the same rough regularity.

**PROBLEMS**

- Differentiability of the classical action.
- Differentiability of the parametrix.
Hamilton’s equations

\[ H(t, x, \xi) = \frac{1}{2}|\xi|^2 + V(t, x), \]
\[ \frac{dx}{dt} = \xi, \quad \frac{d\xi}{dt} = -\nabla_x V(t, x). \] (3)

\[ x(s) = y, \quad \xi(s) = \eta \]

Integral formulation

\[ x(t) = y + \int_s^t \xi(\tau)d\tau \] (4)

\[ \xi(t) = \eta - \int_s^t \nabla_x V(\tau, x(\tau))d\tau. \] (5)
Proposition

For any $t, s \in \mathbb{R}$, the system (3) has a unique solution $x(t) = x(t, s, y, \eta)$ and $\xi(t) = \xi(t, s, y, \eta)$. $x(t)$ is of class $C^1$ in $t$ and $\xi(t)$ is absolutely continuous in $t$.

Sketch (Fixed point iteration)

$$
\Phi(X)(t) = X_s + \int_s^t F(\tau, X(\tau))d\tau, \quad F(t, X) = (\xi, -\nabla_x V(t, x))
$$

$$
|\Phi(X)(t) - \Phi(\tilde{X})(t)| \leq \int_s^t |X(\tau) - \tilde{X}(\tau)|(1 + \|\nabla^2_x V(\tau)\|_{L_x^\infty(\mathbb{R}^n)})d\tau
$$

$$
\leq \sqrt{t-s}(1 + M) \sup_{t \in [s, s+\alpha]} |X(t) - \tilde{X}(t)|.
$$

REMARK ($C^{\frac{1}{2}}$-continuity in time)

$$
|X(t+h) - X(t)| \leq \int_t^{t+h} |F(\tau, x(\tau))|d\tau \leq \sqrt{h}(1 + \|\nabla^2_x V\|_{L_t^2L_x^\infty})
$$
Differentiability w.r.t. initial data

**GOAL** Define the phase function $S(t, s, x, y)$ for the oscillatory integral operator.

We want to write $\eta$ as a function of $(t, s, x, y)$. For this reason we shall study derivatives of $x(t)$ and $\xi(t)$ with respect to the initial values $(y, \eta)$.

\[
\frac{\partial x}{\partial u} = \frac{\partial y}{\partial u} + \int_s^t \frac{\partial \xi}{\partial u}(\tau) d\tau,
\]

\[
\frac{\partial \xi}{\partial u} = \frac{\partial \eta}{\partial u} - \int_s^t \frac{\partial^2}{\partial x^2} V(\tau, x(\tau)) \frac{\partial x}{\partial u}(\tau) d\tau
\]

Set $M := \| \partial^2_x V \|_{L^2_t L^\infty_x}$.

$\Rightarrow$ This integral formulation is fully compatible with Fujiwara’s construction when differentiating with respect to initial data.
Differentiability w.r.t. initial data

\[ \left\| \frac{\partial \xi}{\partial \eta}(t) - I \right\|_{L^\infty(\mathbb{R}_y^n \times \mathbb{R}_\eta^n)} \leq MC |t - s|^{3/2}, \quad \left\| \frac{\partial \xi}{\partial y}(t) \right\|_{L^\infty(\mathbb{R}_y^n \times \mathbb{R}_\eta^n)} \leq MC |t - s|^{1/2} \]

\[ \left\| \frac{\partial x}{\partial y} - I \right\|_{L^\infty(\mathbb{R}_y^n \times \mathbb{R}_\eta^n)} \leq MC |t - s|^{3/2}, \quad \left\| \frac{\partial x}{\partial \eta}(t) \right\|_{L^\infty(\mathbb{R}_y^n \times \mathbb{R}_\eta^n)} \leq C |t - s|, \]

\[ \sum_{2 \leq |\alpha| + |\beta| \leq k} \left\{ |t - s|^{-|\beta| - 3/2} \left| \left( \frac{\partial}{\partial y} \right)^\alpha \left( \frac{\partial}{\partial \eta} \right)^\beta x(t, s, y, \eta) \right| + |t - s|^{-|\beta| - 1/2} \left| \left( \frac{\partial}{\partial y} \right)^\alpha \left( \frac{\partial}{\partial \eta} \right)^\beta \xi(t, s, y, \eta) \right| \right\} \leq C_k \]
Invertibility

We introduce the new variable $\zeta = (t - s)\eta$ and we consider

$$(\tilde{x}(t, s, y, \zeta), \tilde{\xi}(t, s, y, \zeta)) := (x(t, s, y, \zeta/(t - s)), (t - s)\xi(t, s, y, \zeta/(t - s)))$$

**Proposition**

There exists $\delta > 0$, depending on $T$, such that for $0 < |t - s| \leq \delta$ and $y \in \mathbb{R}^n$, the map

$$\zeta \mapsto \tilde{x}(t, s, y, \zeta) = x(t, s, y, \zeta/(t - s))$$

is invertible in $\mathbb{R}^n$. 

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\[ \det \frac{\partial \tilde{x}}{\partial \zeta} = \det \left[ I - (t - s)^{\frac{3}{2}} f(t, s, y, \zeta) \right], \]

for some function \( f(t, s, y, \zeta) \), belonging to a bounded set in \( \mathcal{B}(\mathbb{R}^n_y \times \mathbb{R}^n_\zeta) \).

Choose \( \delta \) such that \((t - s)^{\frac{3}{2}} \| f(t, s, \cdot, \cdot) \|_{L^\infty(\mathbb{R}^n_y \times \mathbb{R}^n_\zeta)} \leq \frac{1}{2}\) holds for any \((y, \zeta) \in \mathbb{R}^{2n}\) and \(0 < |t - s| \leq \delta\).

\[ \implies \eta = \eta(t, s, x, y) \]
\[ 0 < t - s \leq \delta. \]

- \( \eta(t, s, x, y) \) is well defined.
- The curve
  \[ \tau \rightarrow x(\tau) = x(\tau, s, y, \eta(t, s, x, y)) \]
  is the unique classical orbit starting from \( y \) at time \( s \) and reaching \( x \) at time \( t \).

**CLASSICAL ACTION**

\[
S(t, s, x, y) = \int_s^t L(\tau, x(\tau), \dot{x}(\tau)) d\tau,
\]
where
\[
L(\tau, x(\tau), \dot{x}(\tau)) = \frac{1}{2} |\dot{x}(\tau)|^2 - V(\tau, x(\tau)).
\]
REMARKS

- $S(t, s, x, y)$ is not locally Lipschitz continuous in $(t, x, y)$, since with respect to time it is just $C^{\frac{1}{2}}$-continuous $\Rightarrow$ No Rademacher’s Theorem.
- $S(t, s, x, y)$ is absolutely continuous in time: fore each $x$ and $y$ there exists a zero measure set $Z_{x, y}$ out of which it is differentiable in time $\Rightarrow \bigcup_{x, y} Z_{x, y} = ?$.

REGULARIZATION OF THE POTENTIAL

$$V_\varepsilon(\cdot, x) = V(\cdot, x) * \rho_\varepsilon, \quad 0 < \varepsilon \leq 1,$$

where $\rho_\varepsilon(t) = \varepsilon^{-1} \rho(\varepsilon^{-1} t)$ is a standard mollifier in $\mathbb{R}$.

- $\| \nabla^2_x V_\varepsilon \|_{L_t^2 L_x^\infty}$ is uniformly bounded with respect to $\varepsilon$. 
Approximate Hamilton’s equations

\[ x_\varepsilon(y) = y + \int_s^t \xi_\varepsilon(\tau) d\tau, \quad \xi_\varepsilon(t) = \eta - \int_s^t \frac{\partial}{\partial x} V_\varepsilon(\tau, x_\varepsilon(\tau)) d\tau. \]

Convergence of the orbits:
For fixed \( s \in \mathbb{R} \), we have that \( x_\varepsilon(t, s, y, \eta) \) converges to \( x(t, s, y, \eta) \) and \( \xi_\varepsilon(t, s, y, \eta) \) converges to \( \xi(t, s, y, \eta) \), uniformly on the compact subset of \( \mathbb{R} \times \mathbb{R}^{2n} \).

Regularized classical action:

\[ S_\varepsilon(t, s, x, y) := \int_s^t L(\tau, x_\varepsilon(\tau), \dot{x}_\varepsilon(\tau)) d\tau. \]

\( S_\varepsilon(t, s, x, y) \to S(t, s, x, y) \) pointwise as \( \varepsilon \to 0 \).
Properties of the action

- **GENERATING FUNCTION**
  
  \[ S(t, s, x, y) \text{ is of class } C^\infty \text{ in } (x, y) \text{ it } t \text{ and } S \text{ are fixed and} \]
  
  \[ \partial_{x_j} S(t, s, x, y) = \xi_j(t, s, y, \eta(t, s, x, y)) \]
  
  \[ \partial_{y_j} S(t, s, x, y) = -\eta_j(t, s, x, y). \]

- **HAMILTON-JACOBI EQUATION**

  \[ \partial_t S(t, s, x, y) + \frac{1}{2} |\nabla_x S(t, s, x, y)|^2 + V(t, x) = 0. \]

- \[ S(t, s, x, y) = \frac{1}{2} \frac{|x-y|^2}{t-s} + \sqrt{t-s} \omega(t, s, x, y), \text{ where} \]

  \[ |\partial_x^\alpha \partial_y^\beta \omega(t, s, x, y)| \leq C_{\alpha\beta}. \]
Parametrices

\[ 0 < t - s \leq \delta \]

\[
E(t, s)\varphi(x) = \left( \frac{1}{2\pi i(t-s)} \right)^{\frac{n}{2}} \int_{\mathbb{R}^n} e^{iS(t,s,x,y)} \varphi(y) dy.
\]

- **Boundedness in** \( L^2 \)
  \[
  \|E(t, s)\varphi\|_{L^2(\mathbb{R}^n)} \leq \gamma_0(\gamma, M)\|\varphi\|_{L^2(\mathbb{R}^n)},
  \]

- \( E(t,s) \) is a continuous mapping of \( W \) into itself, where \( W = H^2 \cap \mathcal{F}(H^2) \).
- \( E(t, s)\varphi \in D(H(t)) \) for almost every \( t \in \mathbb{R} \).
- In \( L^2 \)
  \[
  \lim_{t \to s} E(t, s)\varphi = \varphi.
  \]

- If we set \( E(s, s) = I \), then the correspondence \( (s, t) \mapsto E(t, s)\varphi \) gives a strongly continuous function with values in \( L^2 \).
**PROBLEM** We can not exchange the order of the integral with the differentiation in time. Indeed, the classical action is absolutely continuous with respect to time, for every $x$ and $y$ fixed. So it is differentiable almost everywhere in $t$, but the zero measure set out of which this property holds depends on $x$ and $y$.

\[
\frac{d}{dt} \int_{\mathbb{R}^n} e^{iS(t,s,x,y)} \varphi(y) \, dy \neq \int_{\mathbb{R}^n} \frac{d}{dt} e^{iS(t,s,x,y)} \varphi(y) \, dy
\]
Differentiation in time of the parametrices

**PROBLEM** We cannot exchange the order of the integral with the differentiation in time. Indeed, the classical action is absolutely continuous with respect to time, for every $x$ and $y$ fixed. So it is differentiable almost everywhere in $t$, but the zero measure set out of which this property holds depends on $x$ and $y$.

\[
\frac{d}{dt} \int_{\mathbb{R}^n} e^{iS(t,s,x,y)} \varphi(y) dy \neq \int_{\mathbb{R}^n} \frac{d}{dt} e^{iS(t,s,x,y)} \varphi(y) dy
\]

\[\implies\] We exploit the regularization of the potential.

\[
E_\varepsilon(t,s)\varphi(x) = \left( \frac{1}{2\pi i(t - s)} \right)^{\frac{n}{2}} \int_{\mathbb{R}^n} e^{iS_\varepsilon(t,s,x,y)} \varphi(y) dy,
\]

\[E_\varepsilon(t,s)\varphi(x) \to E(t,s)\varphi(x).\]
Approximate solution

\( E(t, s)\varphi(x) \) is an approximation of the solution of (1).

**Theorem**

Let \( \varphi \in C_0^\infty(\mathbb{R}^n) \). Then we have that

\[
\left( i\partial_t + \frac{1}{2}\Delta - V(t, x) \right) E(t, s)\varphi(x) = -G(t, s)\varphi,
\]

in \( \mathcal{D}'([s, t] \times \mathbb{R}^n) \).

**REMAINDER TERM**

\[
G(t, s)\varphi(x) = \frac{i\sqrt{t - s}}{(2\pi i(t - s))^{n/2}} \int_{\mathbb{R}^n} e^{iS(t,s,x,y)} \Delta_x \omega(t, s, x, y) \varphi(y) dy.
\]

\[
\|G(t, s)\varphi\|_{L^2(\mathbb{R}^n)} \leq C \sqrt{t - s} \|\varphi\|_{L^2(\mathbb{R}^n)}.
\]
Construction of the fundamental solution

\{ F(t, s) \mid (t, s) \in [-T, T] \times [-T, T] \} \text{ linear operators in } L^2(\mathbb{R}^n)

- \| F(t, s) \varphi \|_{L^2(\mathbb{R}^n)} \leq e^{C_1 |t-s|\gamma_1} \| \varphi \|_{L^2(\mathbb{R}^n)}, \quad \gamma_1 > 0.

- \|(F(t, s_1)F(s_1, s) - F(t, s))\varphi\|_{L^2} \leq C_2 (|t - s_1|^{\alpha} + |s_1 - s|^{\alpha}) \| \varphi \|_{L^2}, \text{ where } |\alpha| > 1.

- \( F(t, s) \varphi \) is a \( L^2(\mathbb{R}^n) \)-valued strongly continuous function in \((t, s) \in \mathbb{R}^2\) with

\[
\begin{aligned}
F(s, s) \varphi &= \varphi \quad \text{for any } s \in \mathbb{R}, \\
\lim_{t \to s} \| F(t, s) \varphi - \varphi \|_{L^2(\mathbb{R}^n)} &= 0.
\end{aligned}
\]

Let \( \Delta \) be a subdivision of the interval \([s, t]\). We put

\[
F(\Delta | t, s) = F(t, t_{l-1})F(t_{l-1}, t_{l-2}) \cdots F(t_1, s).
\]

\[
\lim_{\omega(\Delta) \to 0} \| U(t, s) - F(\Delta | t, s) \| = 0,
\]

in the norm of bounded operator from \( L^2(\mathbb{R}^n) \) to \( L^2(\mathbb{R}^n) \).
Construction of the fundamental solution

**Theorem 2**

For any $\varphi \in \mathcal{W}$, we have

$$i \frac{\partial}{\partial t} U(t, s) \varphi = \frac{-1}{2} \Delta U(t, s) \varphi + V(t, x) U(t, s) \varphi,$$

in $\mathcal{D}'([s, t] \times \mathbb{R}^n)$.

If $\varphi \in \mathcal{W}$, it follows that $H(t) U(t, s) \varphi \in L^2(\mathbb{R}^n)$ for a.e. $t$. By using Theorem 2, we have that

$$i \frac{\partial}{\partial t} U(t, s) \varphi = H(t) U(t, s) \varphi,$$

in $L^2(\mathbb{R}^n)$ for almost every $t$. 

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Thank you!
\[ \partial_t S_\varepsilon(t, s, x, y) + \frac{1}{2} \left| \nabla_x S_\varepsilon(t, s, x, y) \right|^2 + V_\varepsilon(t, x) = 0. \]

\[ S_\varepsilon(t, s, x, y) + \frac{1}{2} \int_s^t \left| \nabla_x S_\varepsilon(\tau, s, x, y) \right|^2 d\tau + \int_s^t V_\varepsilon(\tau, x) d\tau = 0. \]

\[ \int_s^t \left| \nabla_x S_\varepsilon(\tau, s, x, y) \right|^2 d\tau \rightarrow \int_s^t \left| \nabla_x S(\tau, s, x, y) \right|^2 d\tau, \]
as \( \varepsilon \rightarrow 0 \).

\[ \int_s^t \left| V_\varepsilon(\tau, x) - V(\tau, x) \right| d\tau \leq \int_s^t \left\| V_\varepsilon(\tau) - V(\tau) \right\|_{L^\infty_x(B)} d\tau \]

\[ \leq \sqrt{t - s} \left\| V_\varepsilon - V \right\|_{L^2_t L^\infty_x(B)} . \]

\[ S(t, s, x, y) + \frac{1}{2} \int_s^t \left| \nabla_x S(\tau, s, x, y) \right|^2 d\tau + \int_s^t V(\tau, x) d\tau = 0. \]
For each $\varepsilon > 0$ we have that

$$\left( i \partial_t + \frac{1}{2} \Delta - V_\varepsilon(t, x) \right) E_\varepsilon(t, s) \varphi(x) = -G_\varepsilon(t, s) \varphi.$$ 

$$i \int_{\mathbb{R}^n} E_\varepsilon(t, s) \varphi(x) \lambda(t, x) \, dx - i \int_{\mathbb{R}^n} \varphi(x) \lambda(s, x) \, dx$$

$$- i \int_s^t \int_{\mathbb{R}^n} E_\varepsilon(\tau, s) \varphi(x) \partial_\tau \lambda(\tau, x) \, dx \, d\tau$$

$$= \int_s^t \int_{\mathbb{R}^n} E_\varepsilon(\tau, s) \varphi(x) \left( - \frac{1}{2} \Delta + V_\varepsilon(\tau, x) \right) \, dx \, d\tau$$

$$- \int_s^t \int_{\mathbb{R}^n} G_\varepsilon(\tau, x) \varphi(x) \lambda(\tau, x) \, dx \, d\tau.$$ 

$$E_\varepsilon(t, s) \varphi \rightharpoonup E(t, s) \varphi \quad \text{weakly in } L^2_{t, x}.$$ 

$$V_\varepsilon \to V \quad \text{strongly in } L^2_t L^\infty_{\text{loc}, x}.$$ 

$$G_\varepsilon(t, s) \varphi \to G(t, s) \varphi \quad \text{in } L^2_{t, x}.$$ 

(6)